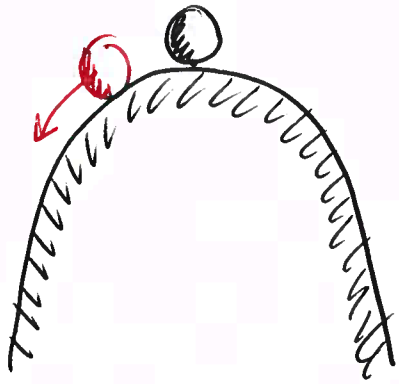
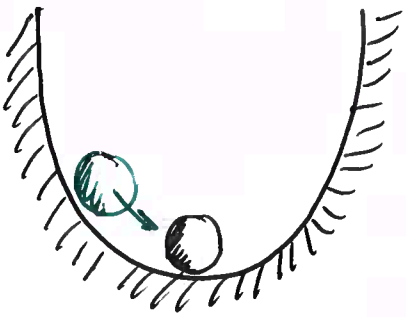
LMB

Lecture 7
13-Oct-2026



UNSTABLE



STABLE

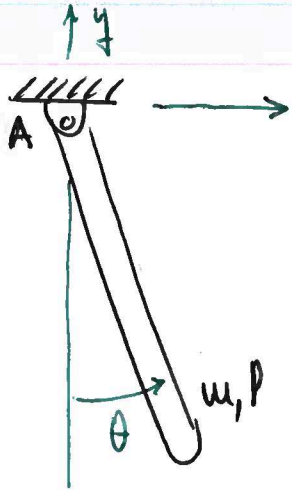
1. Understand behaviour of system (geom., motion, etc.)
2. Generalised coordinates (d.o.f.)
3. Write kinetic energy (T) and potential energy (V)
4. Lagrange equations ⇒ Equations of motion

DON'T LINEARISE

5. Stability analysis: theoretical
   - 5.1 What are stationary positions of system
   - 5.2 Add small perturbation to EoM (eg. δx, $\delta\theta$)
   - 5.3 Linearise EoM
   - 5.4 Write characteristic equation → λ_i
   - 5.5 Analyse λ_i

↓

- $Re = 0$ or $Re < 0$ ⇒ STABLE
- $Re > 0$ ⇒ UNSTABLE



LMB

Lecture 8
16-Oct-2026

$$\begin{cases} T = \frac{1}{6} m l^2 \dot{\theta}^2 \\ V = V_g = -\frac{1}{2} mgl \cos\theta \end{cases}$$

Lecture 3 (29-sep)

EoM (θ): $I_A \ddot{\theta} + \frac{1}{2} mgl \sin\theta = 0$

$$\boxed{\frac{1}{3} m l^2 \ddot{\theta} + \frac{1}{2} mgl \sin\theta = 0}$$

(5.1) Stationary positions $\begin{cases} \theta^s = 0 \\ \theta^s = \pi \end{cases}$ $(\theta^s = n\pi \quad n = 0, 1, 2, \dots)$

(5.2) for $\theta^s = 0$

$$\begin{cases} \theta = \theta^s + \delta\theta = 0 + \delta\theta = \delta\theta \\ \dot{\theta} = \delta\dot{\theta} \\ \ddot{\theta} = \delta\ddot{\theta} \end{cases}$$

EoM $\frac{1}{3} m l^2 \delta\ddot{\theta} + \frac{1}{2} mgl \sin(\delta\theta) = 0$

$\sin(\delta\theta) \approx \delta\theta$

Const. $\to 0$
$\delta\theta \to 1$
$\delta\dot{\theta} \to \lambda$
$\delta\ddot{\theta} \to \lambda^2$

(5.3) $\frac{1}{3} m l^2 \delta\ddot{\theta} + \frac{1}{2} mgl\, \delta\theta = 0$

(5.4) $$\boxed{\frac{1}{3} m l^2 \lambda^2 + \frac{1}{2} mgl = 0}$$ characteristic equation

Solving for $\lambda \Rightarrow \lambda^2 = \left(-\frac{1}{2} mgl\right) \Big/ \left(\frac{1}{3} ml^2\right)$

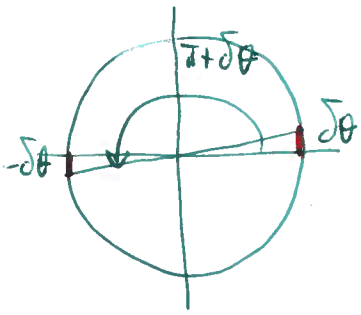
$$\pm i\sqrt{\frac{3g}{2l}} = \lambda = \pm\sqrt{-\frac{3}{2}\frac{g}{l}} \Rightarrow Re = 0$$

($-\frac{3}{2}\frac{g}{l} > 0$)

$\Rightarrow$ STABLE

for $\theta^s = \pi$

$$\begin{cases} \theta = \theta^s + \delta\theta = \pi + \delta\theta \\ \dot\theta = \delta\dot\theta \\ \ddot\theta = \delta\ddot\theta \end{cases}$$



EoM: $\frac{1}{3}ml^2\delta\ddot\theta + \frac{1}{2}mgl\sin(\pi+\delta\theta) = 0$

$\sin(\pi+\delta\theta) \approx -\delta\theta$

Linearise: $\frac{1}{3}ml^2\delta\ddot\theta + \frac{1}{2}mgl(-\delta\theta) = 0$

$$\boxed{\frac{1}{3}ml^2\delta\ddot\theta - \frac{1}{2}mgl\,\delta\theta = 0}$$

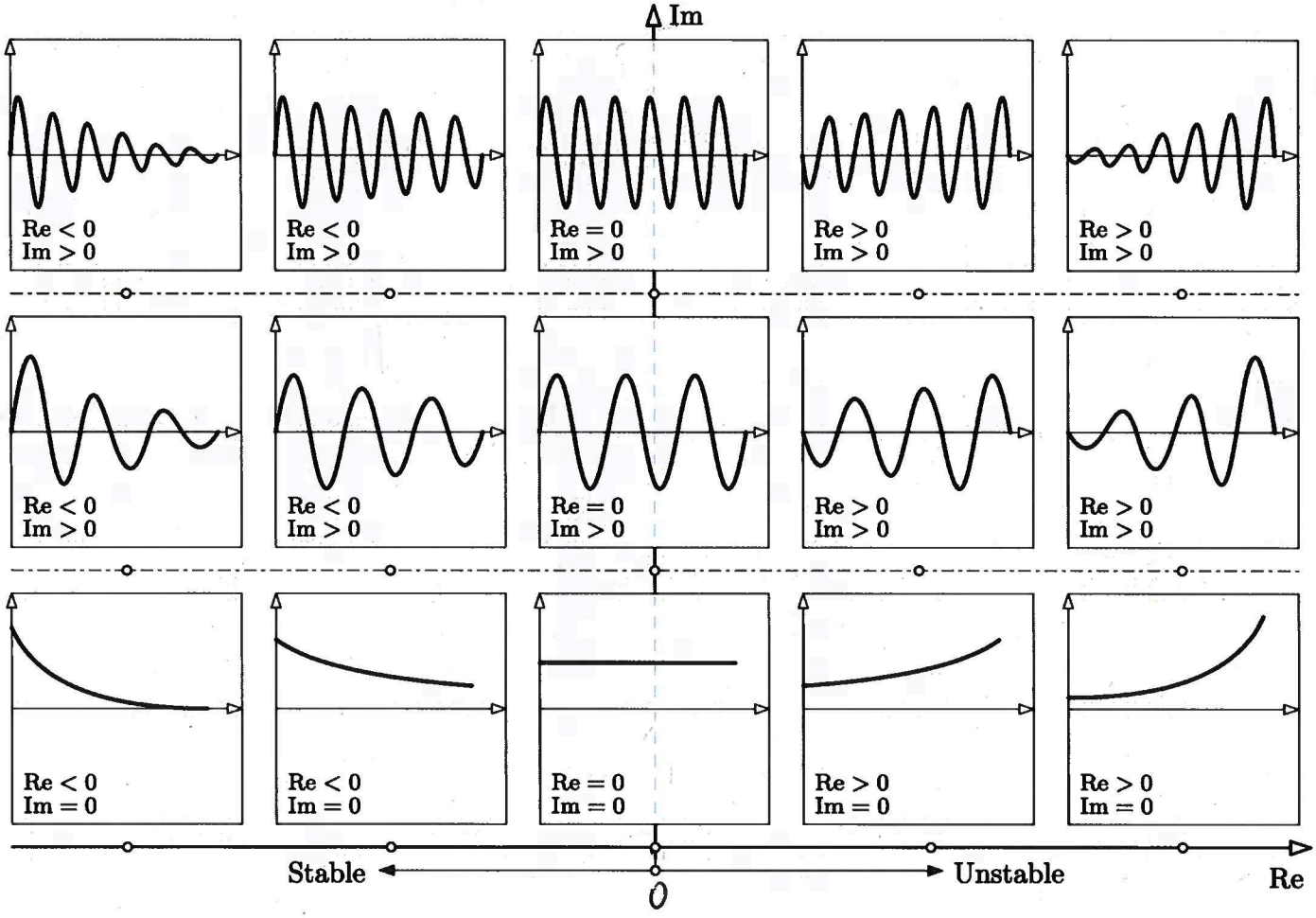
Characteristic equation

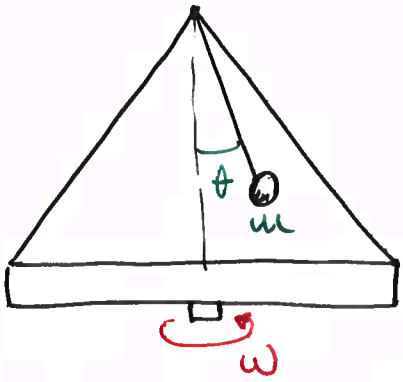
$$\frac{1}{3}ml^2\lambda^2 - \frac{1}{2}mgl = 0$$

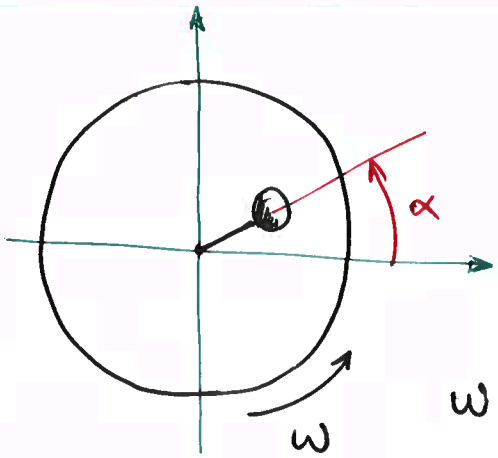
$$\Rightarrow \lambda = \pm\sqrt{\frac{3g}{2l}}$$

↳ $Re \neq 0$ and $Re > 0$

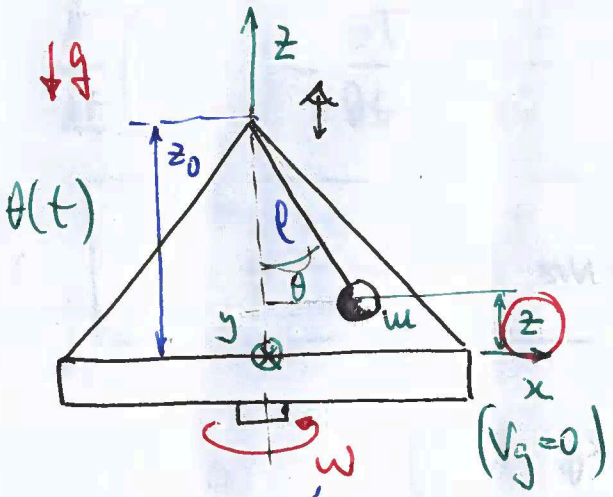
⇒ UNSTABLE

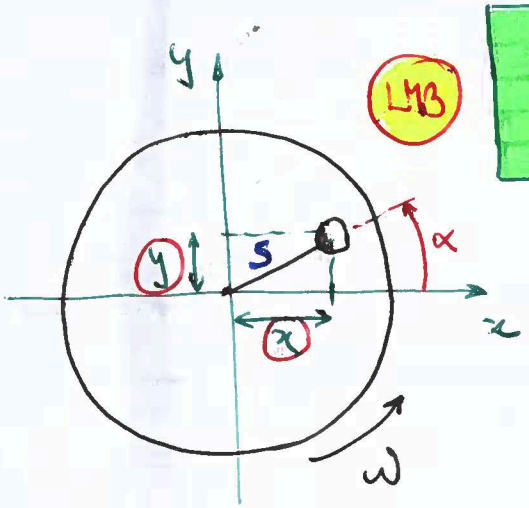

Im
Re < 0
Im > 0
Re < 0
Im > 0
Re = 0
Im > 0
Re > 0
Im > 0
Re > 0
Im > 0
Re < 0
Im > 0
Re < 0
Im > 0
Re = 0
Im > 0
Re > 0
Im > 0
Re > 0
Im > 0
Re < 0
Im = 0
Re < 0
Im = 0
Re = 0
Im = 0
Re > 0
Im = 0
Re > 0
Im = 0
Stable
Unstable
Re
0


θ
m
ω


α
ω
ω = α̇

Lecture 9
20-Oct-2026





$s \neq l$

$s = l \sin\theta$

$\omega = \dot{\alpha}$

Boundary motion
(not a generalised coordinate)

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)$$

$$\begin{cases} x = s\cos\alpha \\ y = s\sin\alpha \\ z = z_0 - l\cos\theta \end{cases} \qquad \begin{cases} x = l\sin\theta\cos\alpha \\ y = l\sin\theta\sin\alpha \\ z = z_0 - l\cos\theta \end{cases}$$

$$\begin{cases} \dot{x} = l\dot{\theta}\cos\theta\cos\alpha - l\sin\theta\,\dot{\alpha}\sin\alpha \\ \dot{y} = l\dot{\theta}\cos\theta\sin\alpha + l\sin\theta\,\dot{\alpha}\cos\alpha \\ \dot{z} = l\dot{\theta}\sin\theta \end{cases} \qquad (\dots)$$

$$\boxed{T = \frac{1}{2} m l^2 \left(\dot{\theta}^2 + \omega^2 \sin^2\theta \right)}$$

$$V = \cancel{V_e} + V_g = mgz = \boxed{mg\left(z_0 - l\cos\theta\right) = V}$$

$V_e = 0$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = \cancel{Q} = 0$$

$$\boxed{m l^2 \ddot\theta - m l^2 \omega^2 \sin\theta\cos\theta + g m l \sin\theta = 0}$$

$\theta^s = 0 \qquad (\text{not } \theta^s = \pi)$

$$\begin{cases} \theta = \cancel{\theta^s}^{\,0} + \delta\theta = \delta\theta \\ \dot\theta = \delta\dot\theta \\ \ddot\theta = \delta\ddot\theta \end{cases}$$

$$m l^2 \delta\ddot\theta - m l^2 \omega^2 \cancel{\sin\delta\theta}^{\,\approx\delta\theta}\ \cancel{\cos\delta\theta}^{\,\approx 1} + m g l \cancel{\sin\delta\theta}^{\,\approx\delta\theta} = 0$$

$$m l^2 \delta\ddot\theta - m l^2 \omega^2 \delta\theta + m g l \delta\theta = 0$$

$$\left(m l^2 \lambda^2 - m l^2 \omega^2 + m g l = 0\right) \Big/ m l^2$$

$$\begin{cases} \text{const} \to 0 \\ \delta\theta \to 1 \\ \delta\dot\theta \to \lambda \\ \delta\ddot\theta \to \lambda^2 \end{cases}$$

$$\lambda^2 - \omega^2 + \frac{g}{l} = 0$$

$$\lambda^2 = \omega^2 - \frac{g}{l} \quad \Rightarrow \quad \lambda = \pm\sqrt{\omega^2 - \frac{g}{l}}$$

$\omega^2 - \frac{g}{\ell} < 0 \Rightarrow \lambda = \pm\sqrt{\omega^2 - \frac{g}{\ell}}$ — Marginally stable

Im

$\omega^2 = \frac{g}{\ell} \Rightarrow \lambda = 0$ — stable

$\omega^2 - \frac{g}{\ell} > 0 \Rightarrow \lambda = \pm\sqrt{\omega^2 - \frac{g}{\ell}}$ — unstable

Re